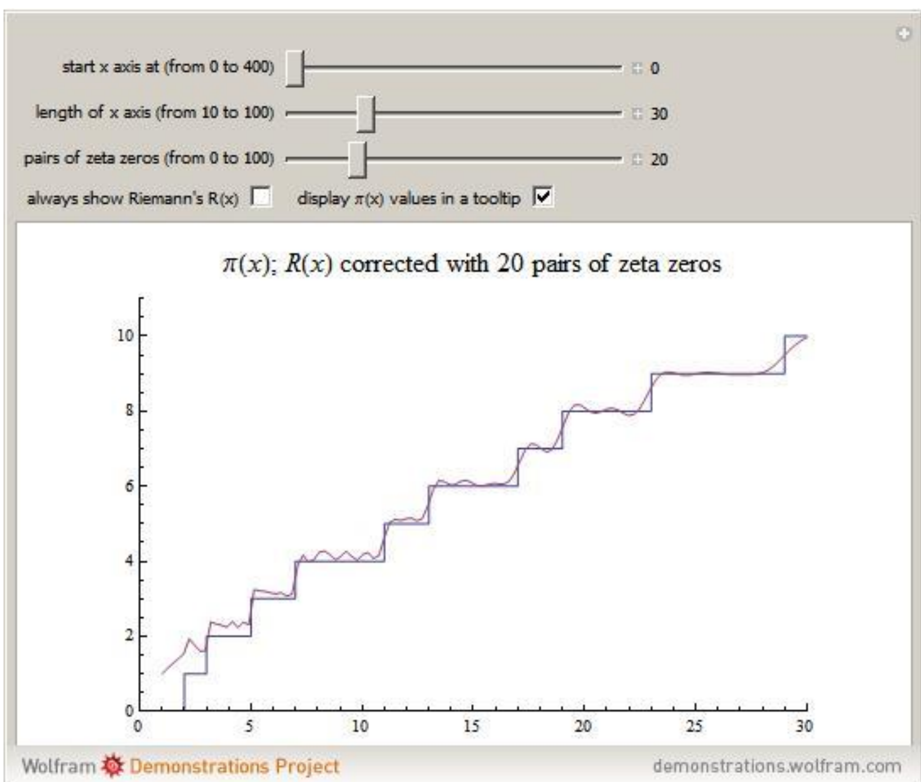
On Number Theory and Quantum Mechanics

Jeremy L. Ebert

ABSTRACT. Physicists use the eigenvalue's of large random matrices to obtain estimates of the average spacing between consecutive energy levels of heavy atomic nuclei and other complex quantum systems. One connection between number theory and quantum mechanics comes from the discovery that these spacing's appear to behave statistically like the spacing's between consecutive zeros of the zeta function. In a similar vein this note sets out to show a link between quantized angular momentum and the divisor summatory function.

1. INTRODUCTION

The zeta function of a real argument was introduced by Leonhard Euler in the first half of the eighteenth century. Bernhard Riemann extended Euler's definition to include a complex argument and established a relation between its zeros and the distribution of prime numbers. The zeros of zeta function can be used to closely track the jumps and irregularities between the prime counting function and the asymptotic law of distribution of prime numbers. Wolfram.com has a beautiful demonstration of this.



http://demonstrations.wolfram.com/HowTheZerosOfTheZetaFunctionPredictTheDistributionOfPrimes/

The divisor summatory function, is a summation over the divisor function. It is frequently used in the study of the asymptotic behavior of the zeta function. The divisor function $d(n)$ counts the number of ways that an integer n can be written as a product of two integers.

- $n = ab = ba$
- $11 = 1 \times 11 = 11 \times 1$
- $d(11) = 2$

For primes the divisor function equals 2, obviously counting the prime and the integer 1. One could think of the prime counting function as only counting the "2's" of the divisor function.

For composite integers, the divisor function is always greater than 2. Prime squares have 3 divisors; in fact, only squares have an odd number of divisors.

- $4 = 1 \times 4 = 4 \times 1 = 2 \times 2$
- $d(4) = 3$
- $9 = 1 \times 9 = 9 \times 1 = 3 \times 3$
- $d(9) = 3$
- $16 = 1 \times 16 = 16 \times 1 = 2 \times 8 = 8 \times 2 = 4 \times 4$
- $d(16) = 5$
- $25 = 1 \times 25 = 25 \times 1 = 5 \times 5$
- $d(25) = 3$

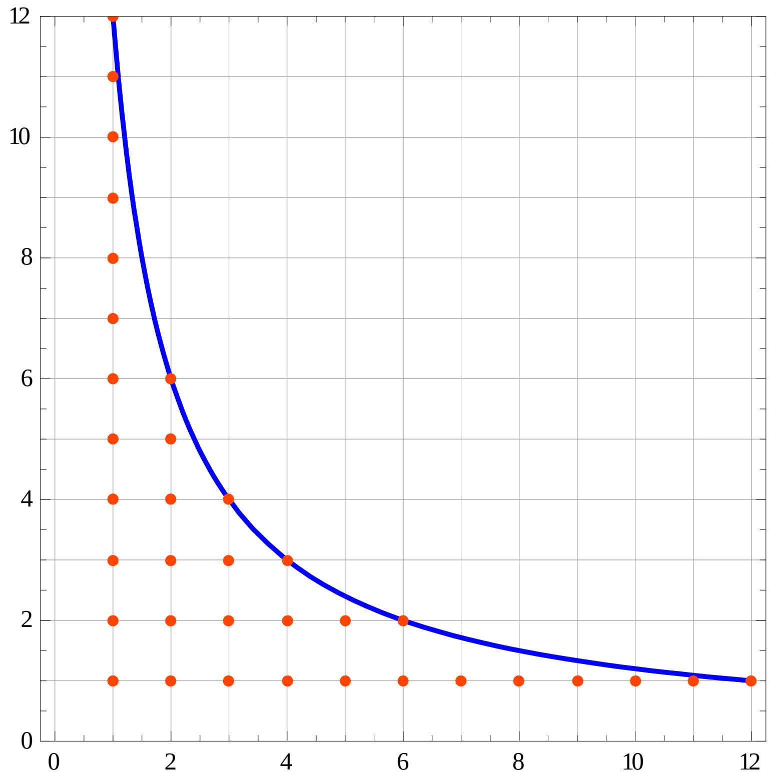
Its trivial to see a prime p to the exponent m will have $m+1$ divisors. d(2^2)=3, d(2^3)=4, d(2^4)=5, d(2^5)=6, d(2^6)=7…

This attribute of square numbers, highlights a trivial but useful property about divisors in general; all divisors below the square root of a number produce a quotient greater than the square root and vice versa, if a divisor is equal to the square root the quotient is equal to the divisor, so the divisor only gets counted once, producing an odd number of divisors.

2. Dirichlet's Estimate

Gustav Lejeune Dirichlet was a German mathematician with deep contributions to number theory. He noticed how divisors and the geometry of conics were related. One contribution he made is a way to estimate the number of lattice points under a hyperbola which is in essence an estimate of the divisor summatory function. The tools he used for this were the hyperbolic logarithm and the Euler-Mascheroni constant.

lattice points under hyperbola : 35
number according to Dirichlet 's estimate : 31.67



http://demonstrations.wolfram.com/LatticePointsUnderAHyperbola/

3. Hyperbolic Logarithm

The natural logarithm was once called the hyperbolic logarithm as it corresponds to the area under the hyperbola created by the function $y = 1/x$. It is the logarithm to the base e. Where e is an irrational and transcendental constant approximately equal to 2.7182818282... It is generally denoted $\ln(n)$. The natural logarithm of a number n is the power to which e would have to be raised to equal n. For example $\ln(7.389\ldots) = 2$ because $e^2 = 7.389\ldots$ and $\ln(e) = 1$ because $e^1 = e$. Euler defined the inverse relation between the exponential function e^n and the hyperbolic logarithm by demonstrating that $\ln(e^n) = n$. **Figure 1** gives the example of $\ln(n)$ where $n = \sqrt{12}$ and γ is the Euler-Mascheroni constant, which is the limiting difference between the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots$ and the natural logarithm, approximately equal to 0.5772156649.

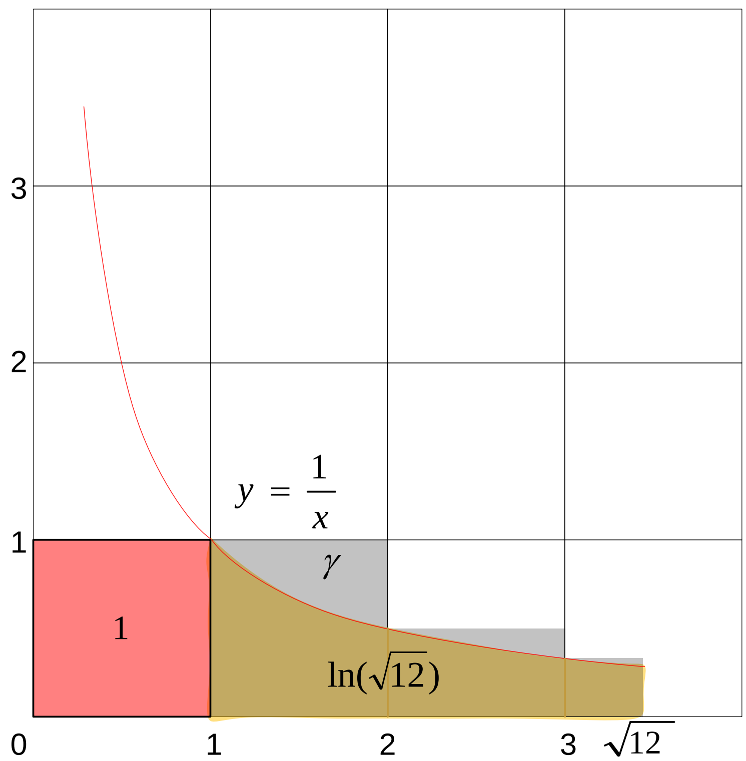
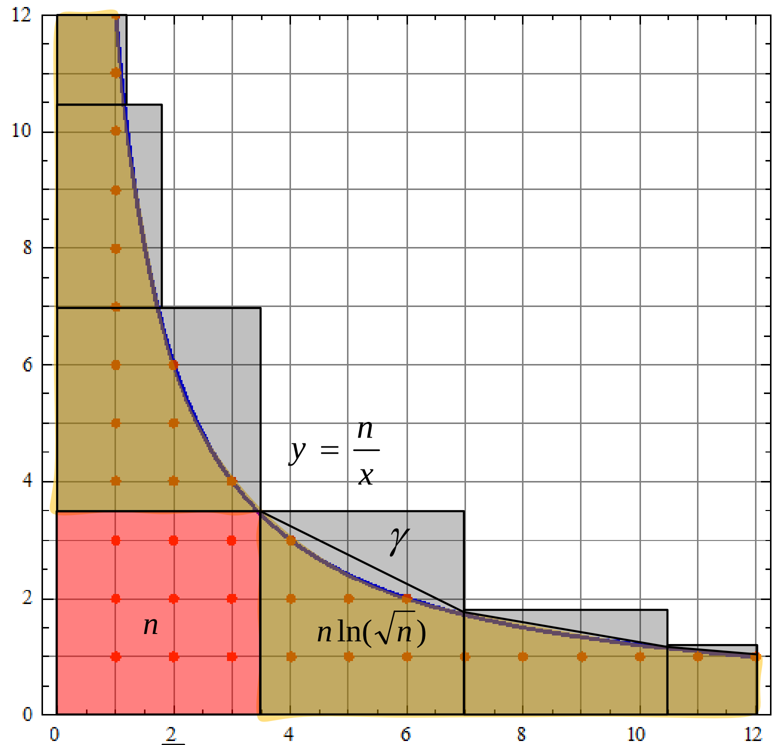


Figure 1. *The natural logarithm as it corresponds to the area under a hyperbola*.

A self similar symmetry can be seen with the natural logarithm and the square root function. This symmetry plays a role in Dirichlet's estimate for the number of lattice points under the hyperbola. See **Figure 2.**

lattice points under hyperbola : 35
number according to Dirichlet 's estimate : 31.67



$2n\ln(\sqrt{n}) = n\ln(n)$ $\quad$ $Estimate = n\ln(n) + (2\gamma - 1)n$

Figure 2. Dirichlet's estimate for the number of lattice points under the hyperbola $xy = n$ in the first quadrant is given by $n\ln(n) + (2\gamma - 1)n$.

$$\frac{n\ln(n)}{2} = n\ln(\sqrt{n}) \qquad \frac{n\ln(n)}{s} = n\ln\left(\sqrt[s]{n}\right) \qquad e^{\frac{\frac{n\ln(n)}{s}}{n}} = n^{\frac{1}{s}} = \sqrt[s]{n}$$

$$e^{\frac{s(n\ln(n))}{n}} = n^s \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

3. Geometric Mean Theorem

The Geometric Mean ($G = \sqrt[n]{x_1 x_2 \cdots x_n}$) is one of the three Pythagorean Means. Geometric Mean Theorem is a result in elementary geometry that describes a relation between the altitude in a right triangle and the two line segments it creates on the hypotenuse. It states that the geometric mean of the two segments equals the altitude. A multiplication table can be derived from these two line segments for a *given hypotenuse*. This table is also called the Table of Pythagoras and is attributed to him by some authors. See **Figure 3.**

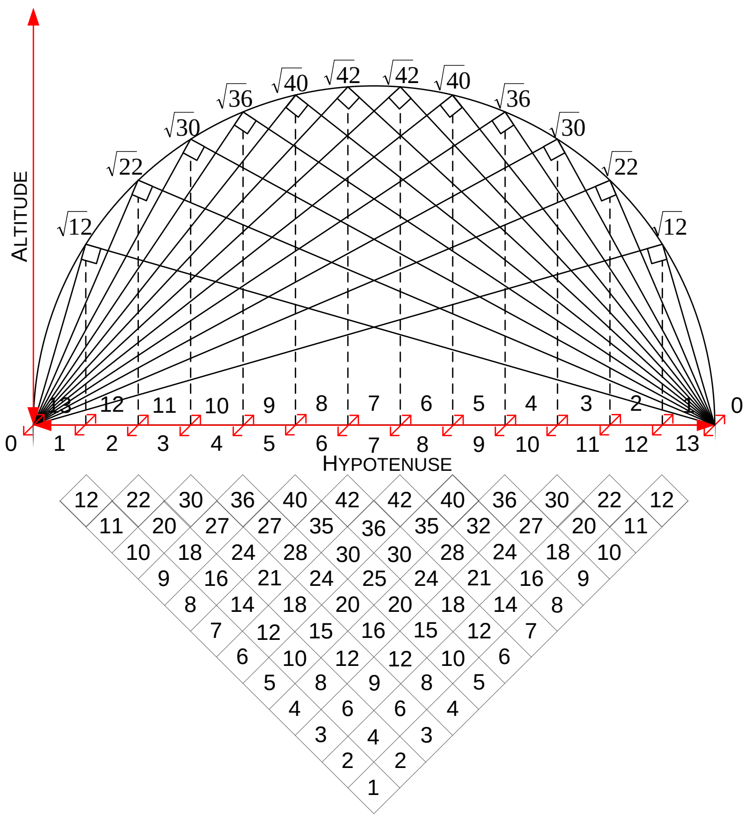


Figure 3. *Geometric Mean Theorem for a given hypotenuse (top). Table of Pythagoras (bottom).*

http://betterexplained.com/articles/understanding-why-complex-multiplication-works/

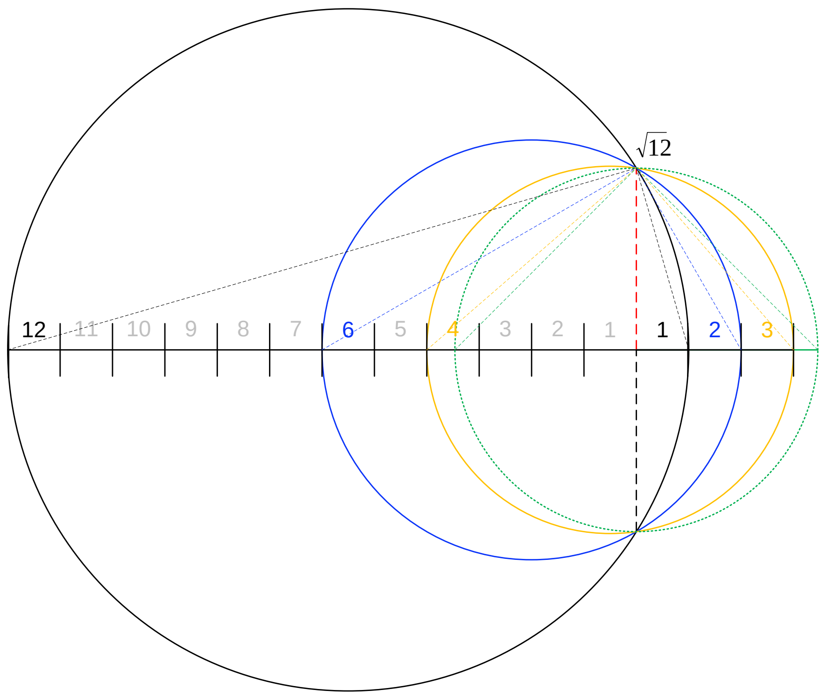


Figure 4. *Divisor and quotient results of the Geometric Mean Theorem for a given altitude*

Another aspect of the Geometric Mean Theorem which directly relates to the divisor function is the two integer valued line segments of a hypotenuse for a *given altitude* is equivalent to a divisor and quotient of the square of the altitude. See **Figure 4**. So from the Geometric Mean Theorem we can geometrically derive the multiplication table by use of the hypotenuse and the divisor function by use of the altitude.

4. Arithmetic-Geometric Mean (AGM)

In mathematics, the inequality of arithmetic ($A = \frac{1}{n}(x_1 + x_2 + \cdots x_n)$) and geometric ($G = \sqrt[n]{x_1 x_2 \cdots x_n}$) means, or more briefly the AM–GM inequality, states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list; and further, that the two means are equal if and only if every number in the list is the same. Arithmetic-Geometric mean type approaches are used as the foundation for fast computational modules in some mathematical packages. For instance here is one approach. (*AGM Method pi algorithm)

From **Figure 4** we can look at the divisors (x_1) and quotients (x_2) of the number 12 and their relation to its Geometric Mean (altitude):

$$G = \sqrt[2]{1 \times 12} = \sqrt[2]{2 \times 6} = \sqrt[2]{3 \times 4} \approx 3.4641016151377545870548926830117$$

Applying the Arithmetic Mean (hypotenuse) we can see the convergence to the Geometric Mean taking place:

$$A = \frac{1}{2}(1 + 12) = 6.5$$

$$A = \frac{1}{2}(2 + 6) = 4$$

$$A = \frac{1}{2}(3 + 4) = 3.5$$

Utilizing the AM–GM inequality to concentrically align these two means results in a projection of the hyperbolic logarithm revealing the true nature of the hyperbola, a conic section. See **Figure 5 a, b, c**.

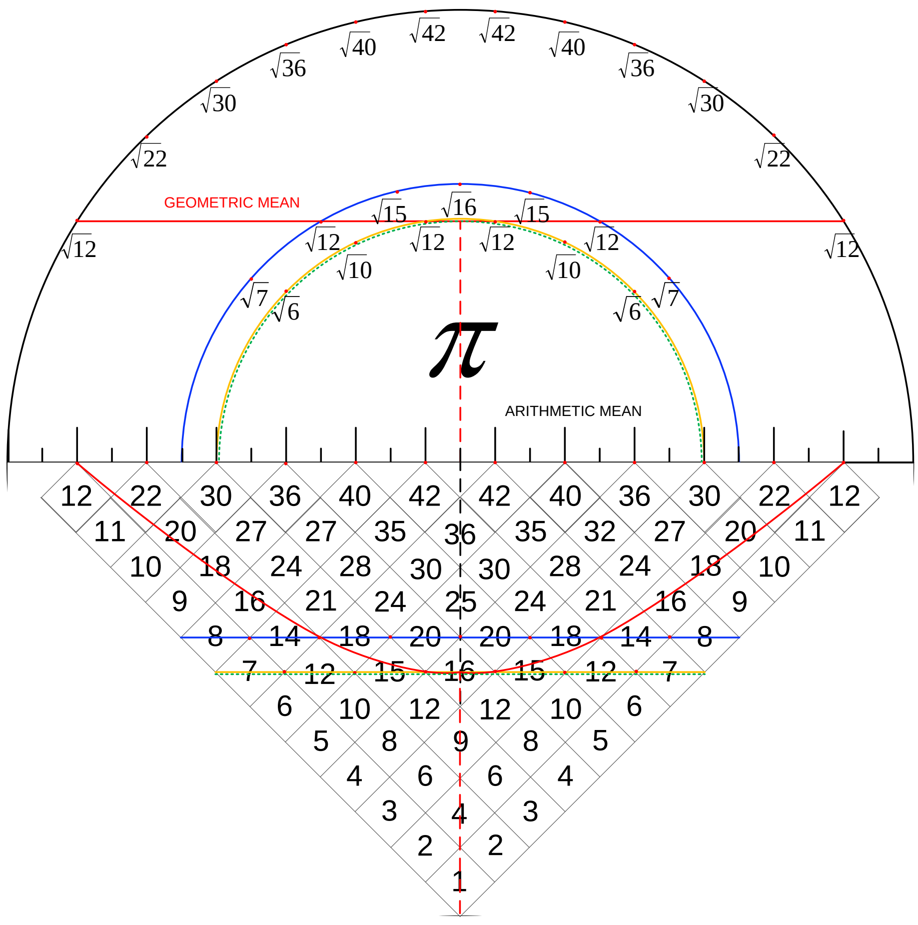


Figure 5 (a). AM–GM symmetry with the hyperbolic logarithm.

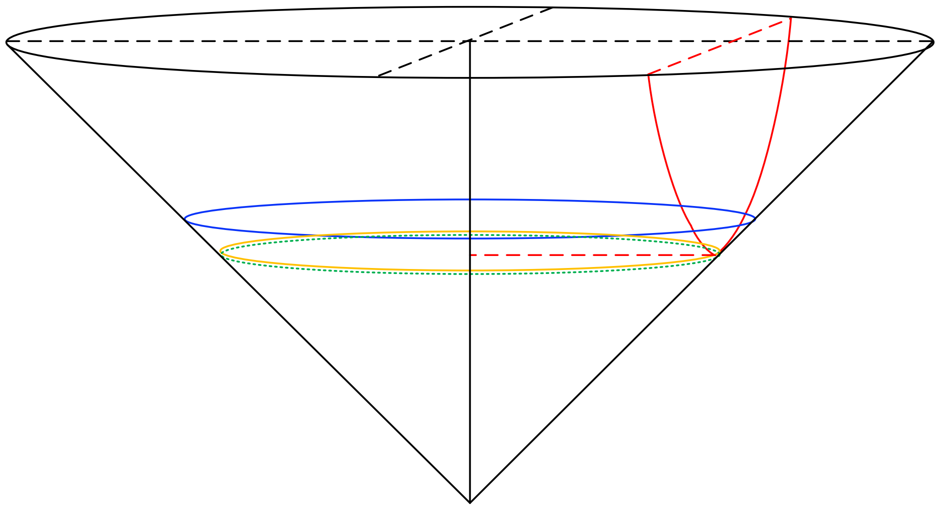


Figure 5 (b). the hyperbola as a conic section

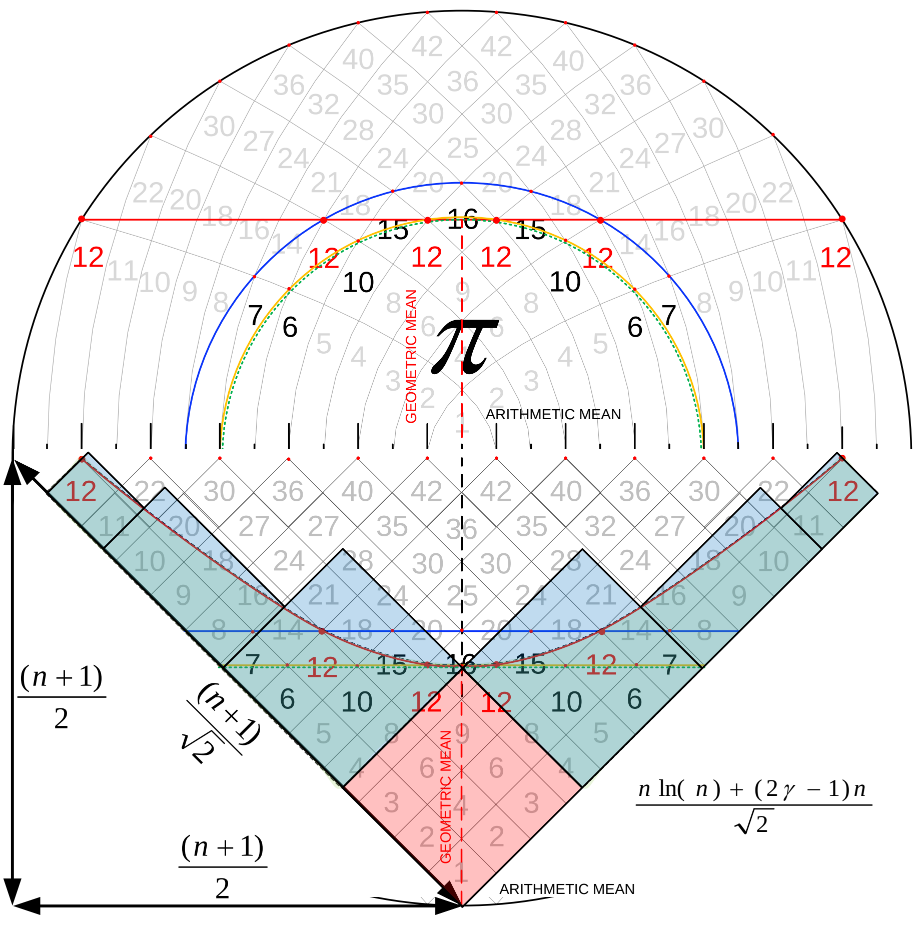


Figure 5(c). *If we view this geometry concentrically it becomes apparent that the square root function and the divisor function are intrinsically related to conics.*

This geometry shows a deep connection with the irrationality of π and square root of a number. We will show later how the ladder operator values of quantized angular momentum mimic the values of this multiplication table.

5. Angular Momentum

A classical example of angular momentum is that of a spinning figure skater. Conservation of angular momentum is demonstrated when she reduces her moment of inertia by pulling in her arms, causing her angular velocity, usually denoted $\boldsymbol{\omega}$, to increase. The moment of the inertial force on a particle around an axis multiplies the mass of the particle by the square of its distance to the axis (think inverse square law), and forms a parameter called the moment of inertia usually denoted $I = mr^2$. In a two dimensional planar motion (think pendulum) this is a scalar quantity, in 3 dimensions it is known as a tensor because direction becomes a factor. Angular momentum is usually denoted as $\mathbf{L}$ giving the equation $\mathbf{L} = I\boldsymbol{\omega}$. Linear momentum, usually denoted $\mathbf{p}$, is the product of the mass and velocity giving the equation $\mathbf{p} = m\mathbf{v}$. Angular momentum can also be expressed as the cross product of linear momentum and its position $\mathbf{r}$ from the axis of rotation, giving the equation $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. See **Figure 6.**

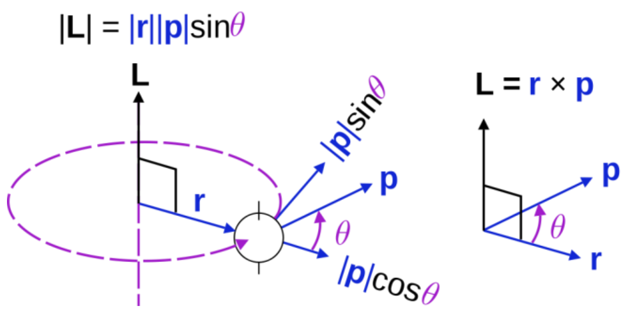


Figure 6. Angular momentum as the cross product of position and linear momentum.

6. Quantum Angular Momentum

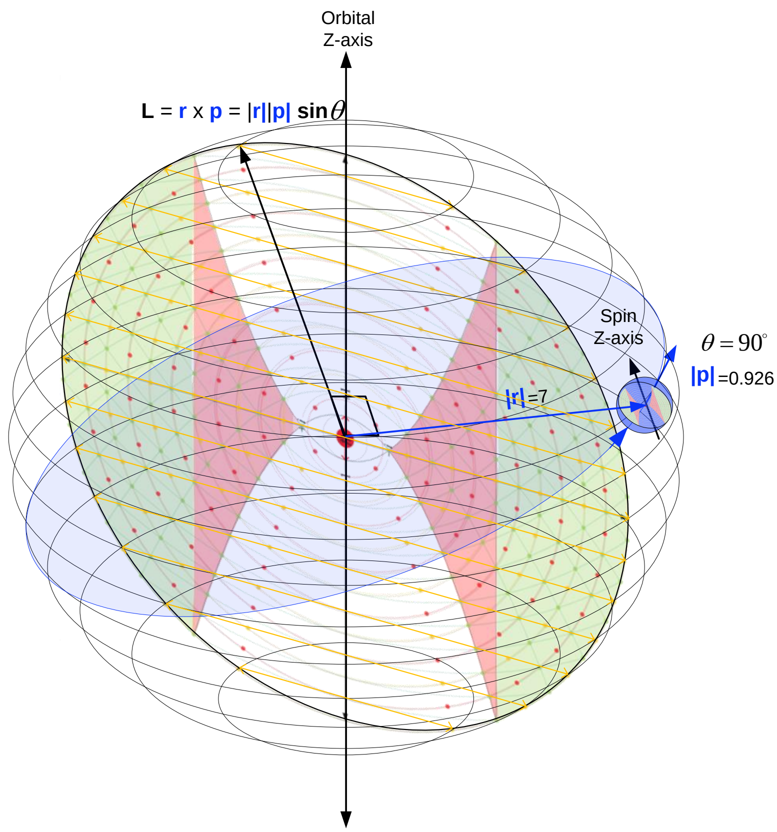
The ideas in classical mechanics can be carried over to quantum mechanics, by reinterpreting $\mathbf{r}$ as the quantum position operator and $\mathbf{p}$ as the quantum momentum operator. $\mathbf{L}$ is then the orbital angular operator. In quantum mechanics there is another angular momentum operator $\mathbf{S}$, know as spin. It can generally be thought of as the earth (electron) spinning ($\mathbf{S}$) on its axis, as it orbits ($\mathbf{L}$) the sun (nucleus), but this a very limited analogy. The total angular momentum ($\mathbf{J}$) of a particle then is given by $\mathbf{J} = \mathbf{L} + \mathbf{S}$. This not a continuous valued function as in classical mechanics, rather it is quantized by integer and half-integer values of the reduced Planck constant $\hbar$. The product of the Planck constant (h) and the frequency (f) of a particle can be used to describe its energy, giving the equation $\mathrm{E} = hf$, reducing the constant , $\hbar = \frac{\boldsymbol{h}}{\boldsymbol{2\pi}}$ gives the equation $\mathrm{E} = \hbar|\boldsymbol{\omega}|$, where the magnitude of its angular velocity $|\boldsymbol{\omega}|$ is its angular frequency.

The Heisenberg uncertainty principle tells us that it is not possible to know a particles exact position $(\mathbf{r})$ and momentum $(\mathbf{p})$ simultaneously. The more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa. It turns out that the best that one can do is to simultaneously measure both the angular momentums magnitude (absolute value $|\mathbf{L}|$) and its component along one axis(L_z). The z-axis is typically used for convention. The math behind the different angular momentum operators (spin $\mathbf{S}$, orbital $\mathbf{L}$ and total $\mathbf{J}$) are carbon copies of each other.

A model used in Quantum Physics describes electrons using four quantum numbers n, l, m_l and m_s.

1. (n) The principal quantum number n describes the electron shell or energy level of an atom. This quantum number is limited to integer values only.
2. (l) The orbital quantum number l describes the subshell and gives the magnitude of the orbital angular momentum $|\mathbf{L}|$ through the relation $|\mathbf{L}| = \sqrt{l(l+1)}\,\hbar$. This quantum number is limited to integer values less than n. $0 \leq l \leq (n-1)$.
3. (m_l) The magnetic quantum number m_l describes the specific orbital (or "cloud") within that subshell, and projects a component of the orbital angular momentum on a specific axis L_z. $L_z = -l \leq m_l \leq +l$. Again, the z-axis is typically used for convention.
4. (m_s) The spin projection quantum number m_s yields the component of the spin angular momentum on a specific axis S_z. $S_z = -s \leq m_s \leq +s$ where s is the spin quantum number and the spin magnitude $|\mathbf{S}| = \sqrt{s(s+1)}\,\hbar$. The quantum number s can be both integer and half-integer values. For electrons, $s = 1/2$ and $ms = -\frac{1}{2},\ +\frac{1}{2}$

There is one more operator that we will consider; it is called a ladder operator. This operator is used to raise and lower the m_s or m_l values by $\hbar$. It is usually denoted as $S_\pm$ for spin and $L_\pm$ for orbital angular momentum. The equation for the ladder operator for orbital angular momentum is $L_\pm = \sqrt{l(l+1) - m_l(m_l \pm 1)}\,\hbar$, as mentioned already, the math is the same for all forms of angular momentum. This operator allows us to step through all of the observable states of angular momentum, with a given magnitude, on a specific axis. See **Figure 7.**



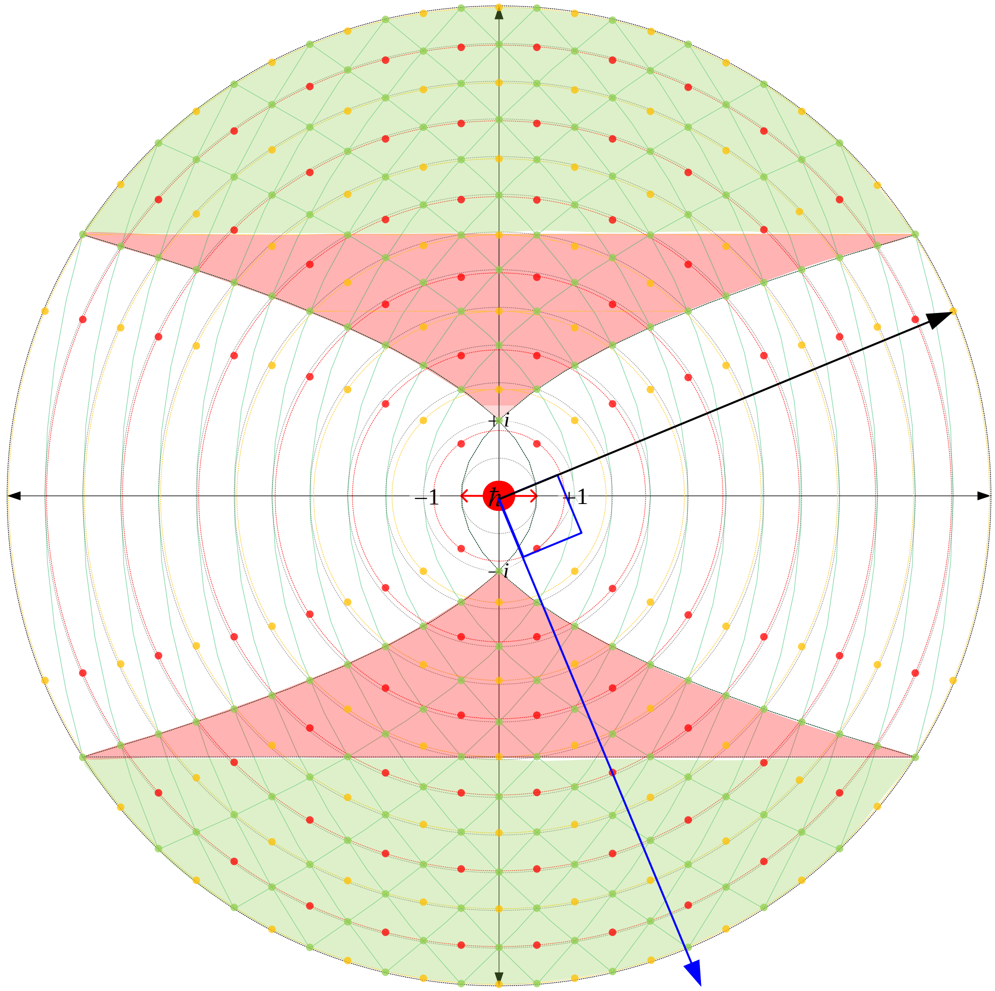
- $n = 7$ $|\mathbf{r}| = 7$ $|\mathbf{p}| = 0.9258$ $\sin\theta = 1$
- $l = 6$ $|\mathbf{L}| = \sqrt{l(l+1)}\,\hbar$ $|\mathbf{L}| = \sqrt{42}\,\hbar$ $|\mathbf{L}| = 6.48074\,\hbar$
- $m_l = \{+6, +5, +4, +3, +2, +1, \sim 0, -1, -2, -3, -4, -5, -6\}$

Figure 7. The three quantum numbers n, l, m_l are used to describe orbital angular momentum. Zeros of the wave function:

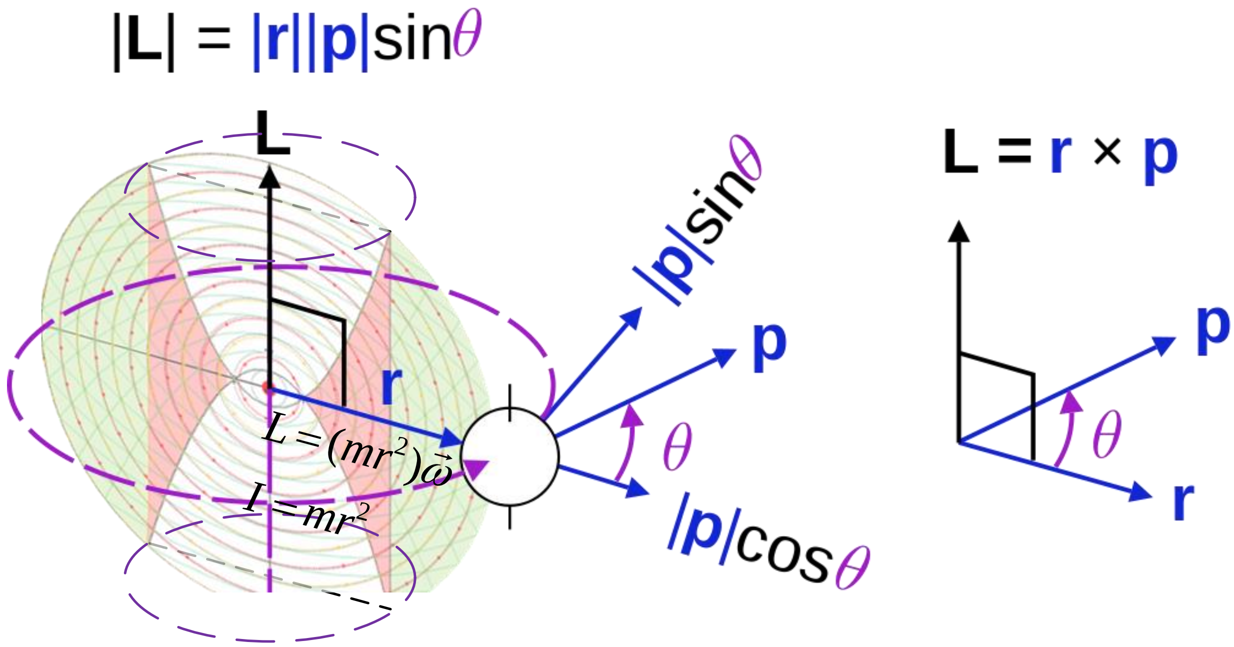
- $n = 6.5$ $|\mathbf{r}| = 6.5$ $|\mathbf{p}| = 1$ $\sin\theta = 1$
- $l = 5.5$ $|\mathbf{L}| = l + 1\,\hbar$ $|\mathbf{L}| = \sqrt{42.25}\,\hbar$ $|\mathbf{L}| = 6.5\,\hbar$
- $m_l = \{+5.5, +4.5, +3.5, +2.5, +1.5, +0.5, \sim 0, -0.5, -1.5, -2.5, -3.5, -4.5, -5.5\}$

NOTES

http://en.wikipedia.org/wiki/Steradian



+i
−1
+1
−i
h



Appling this information along with the Heisenberg uncertainty principle to the generalized angular momentum operator it holds that we can simultaneously measure only two components of each type of angular momentum:

$$|\boldsymbol{A}| = \sqrt{a(a+1)}\,\hbar$$

and

$$A_z = m_a\,\hbar$$

where

- $\boldsymbol{A}$ is the quantized angular momentum vector.
- $|\boldsymbol{A}|$ is the norm or magnitude of the vector.
- a is the quantum number associated with the particular type of angular momentum.

- A_i is one component of the angular momentum on an arbitrary axis i. The z-axis is typically used for convention specified by A_z.
- m_a is a projection quantum number and ranges from $-a$ to $+a$ in steps of one. This generates $2a + 1$ different values of m_a.
- ħ is the reduced Planck constant.

A graphical representation of a and m_a is presented in Figure 2. Utilizing the Pythagorean theorem a representation of the observables $|\boldsymbol{A}|$ and A_z is presented in Figure 3.

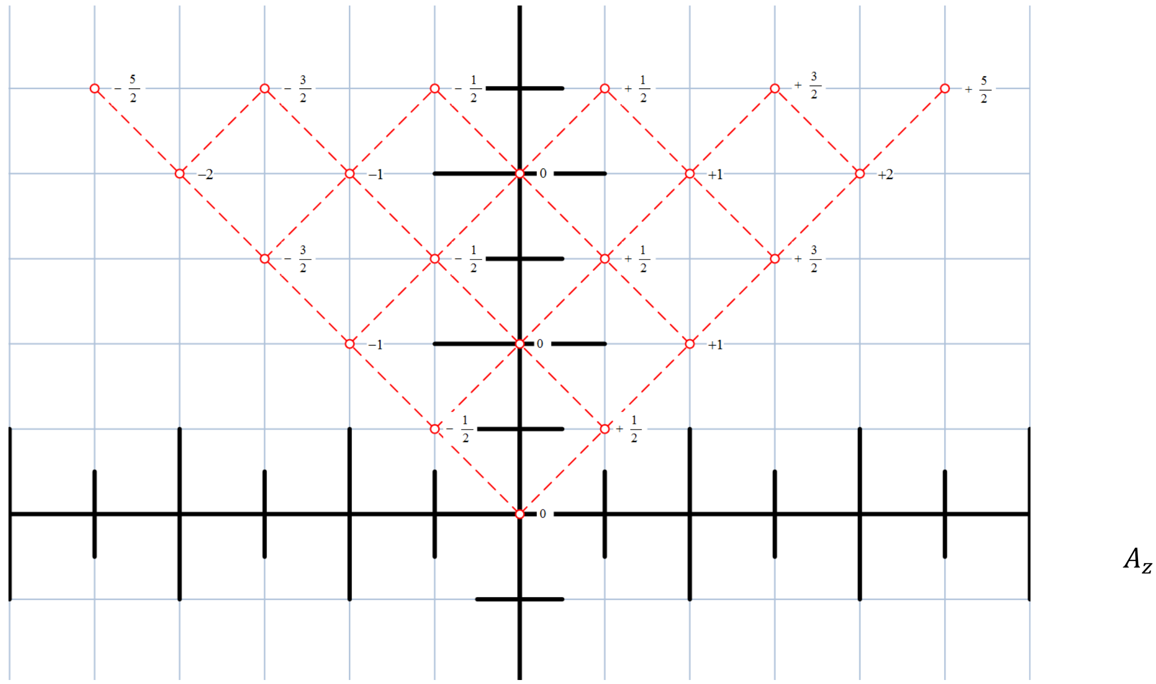


Figure 2. The values of a and m_a can be arranged in a symmetric triangular array to aid in visualizing their values. $a = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\}$

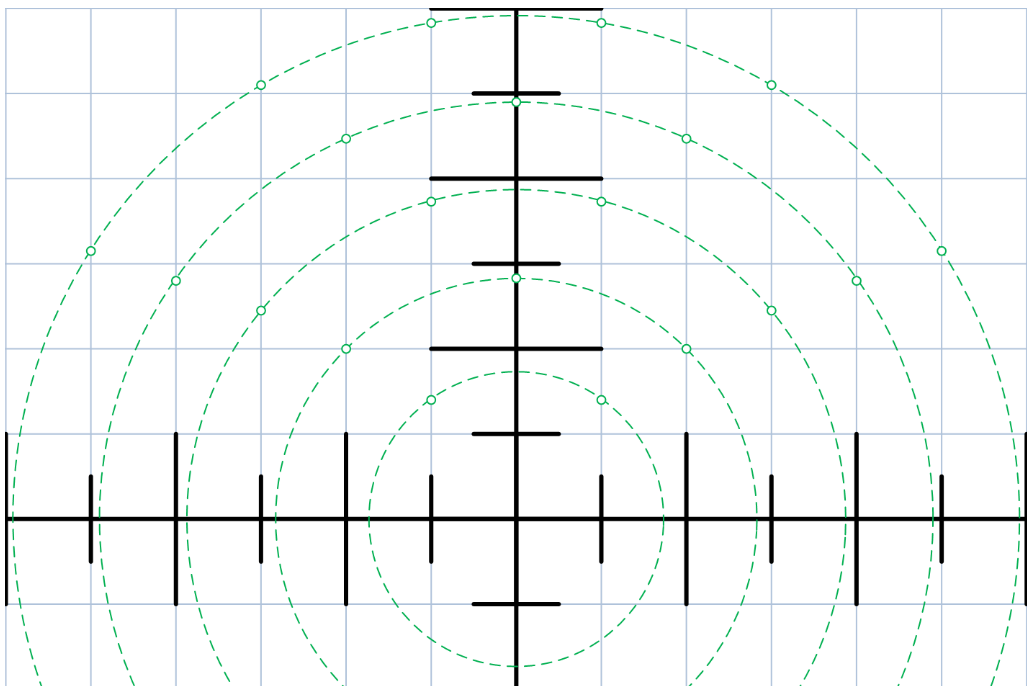


Figure 3. Due to the Heisenberg uncertainty principle, the observable values of an angular momentum vector $\boldsymbol{A}$ are $|\boldsymbol{A}|$ and A_z

The values of a and m_a can be generated with the following basic equations:

- $a = \frac{n+1}{2} - 1$ for an integer n.
- $m_a = \frac{n+1}{2} - k$ where $1 \leq k \leq n$ for an integer k.

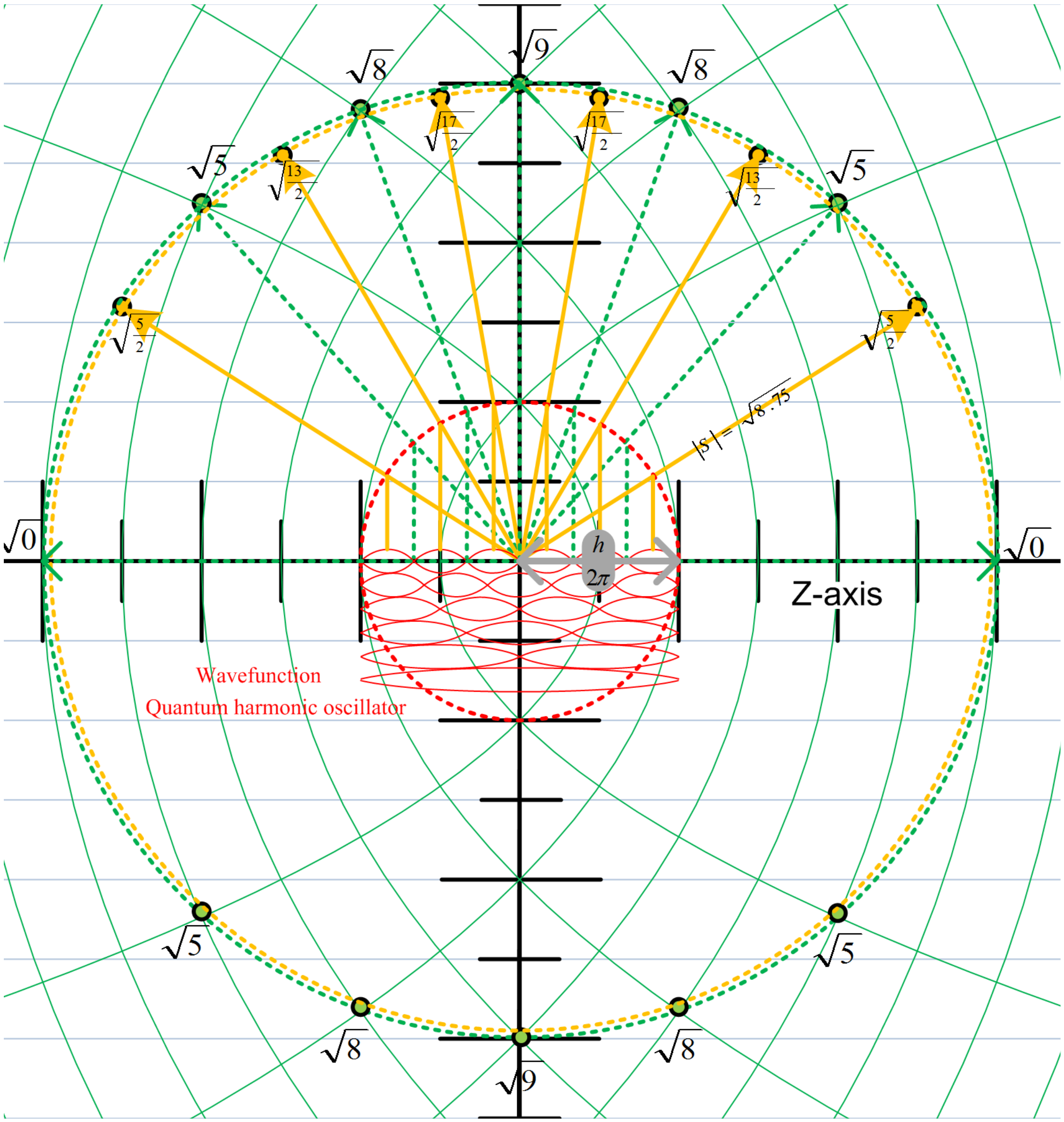
Notice that the using these equations the values of m_a step down from $+a$ to $-a$. In quantum mechanics this would be considered a lowering operator denoted by A_- . It has a companion the raising operator A_+. They are called ladder operators and are typically denoted by $A_\pm$. The values of $A_\pm$ are typically defined as:

- $A_+ = \sqrt{a(a+1) - m_a(m_a+1)}\ \hbar = \sqrt{(k-1)(n+1-k)}\,\hbar$
- $A_- = \sqrt{a(a+1) - m_a(m_a-1)}\ \hbar = \sqrt{k(n-k)}\ \hbar$
- $A_\pm = A_x \pm i\, A_y$

These operators eigenvaulues can be thought of as the relative intensities between the observable states of A_z. If thought of as a vector, the magnitude of $A_\pm$ and the z-axis component $A_{z_\pm}$ could be interpreted as:

- $|A_\pm| = a + \frac{1}{2} = \sqrt{a(a+1) + \frac{1}{4}} = n/2$

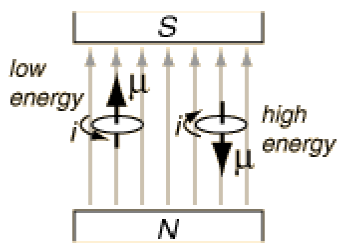
- $A_{z_+} = k - \frac{n}{2}$
- $A_{z_-} = \frac{n}{2} - k$



A magnetic dipole moment in a magnetic field will possess potential energy which depends upon its orientation with respect to the magnetic field. Since magnetic sources are inherently dipole sources which can be visualized as a current loop with current I and area A, the energy is usually expressed in terms of the magnetic dipole moment:

$$U(\theta) = -\mu \cdot B \quad \text{where} \quad \mu = IA$$

The energy is expressed as a scalar product, and implies that the energy is lowest when the magnetic moment is aligned with the magnetic field. The difference in energy between aligned and anti-aligned is

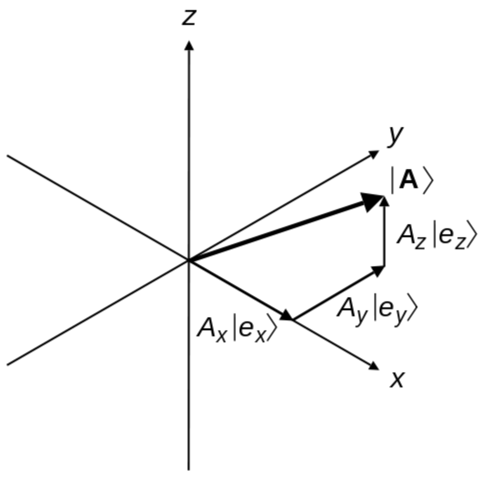


$$\Delta U = 2\mu B$$

The expression for magnetic potential energy can be developed from the expression for the magnetic torque on a current loop.

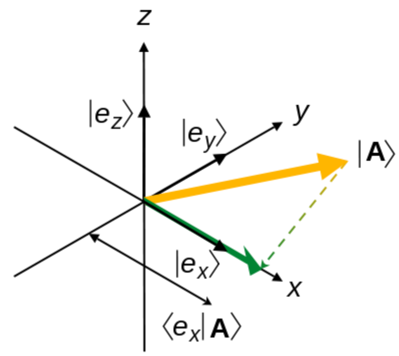
These relationships for a finite current loop extend to the magnetic dipoles of electron orbits and to the intrinsic magnetic moments associated with electron spin and nuclear spin.

Ket



$$\mathbf{A} = A_x|e_x\rangle + A_y|e_y\rangle + A_z|e_z\rangle$$

Bases and components, projections
of the full vector onto the bases



$$\mathbf{A} = \langle e_x|\mathbf{A}\rangle|e_x\rangle + \langle e_y|\mathbf{A}\rangle|e_y\rangle + \langle e_z|\mathbf{A}\rangle|e_z\rangle$$

$$A_x = \langle e_x|\mathbf{A}\rangle$$

$$A_y = \langle e_y|\mathbf{A}\rangle$$

$$A_z = \langle e_z|\mathbf{A}\rangle$$

a=2.5

n=6

|A|^2 = 8.75

p=(n-1)

A-=sqrt((k(n-k)))=(k(p+1-k))={+5,+8,+9,+8,+5,~0}

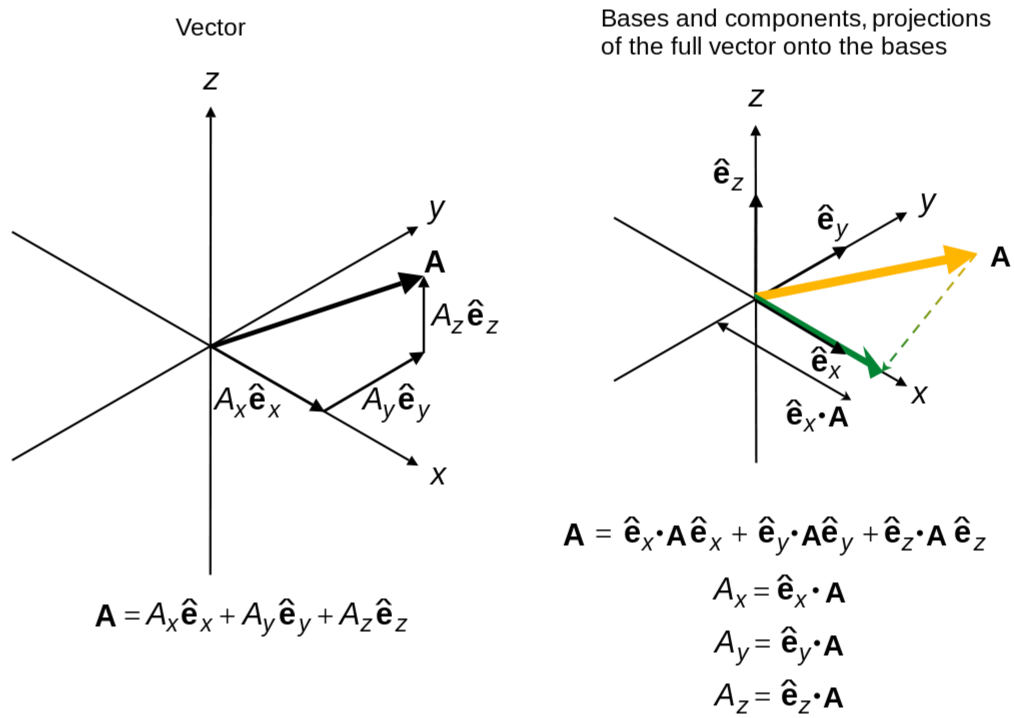
A+=sqrt(((k-1)(n+1-k)))=((k-1)(p+2-k))={~0,+5,+8,+9,+8,+5}

ma-=(((n-1)+1)/2)-k =(n/2)-k=(p+1/2)-k={+2,+1,~0,-1,-2,-3}

ma+=(((n-1)+1)/2)+k =-(n/2)+k=-(p+1/2)+k={-2,-1,~0,+1,+2+3}

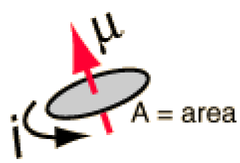
|A-+|=((n-1)+1)/2 = n/2=(p+1/2)=3=radius/magnitude of the vector

|A-+|^2 = 9



Magnetic Dipole Moment

From the expression for the torque on a current loop, the characteristics of the current loop are summarized in its magnetic moment



$$\mu = IA \quad .$$

The magnetic moment can be considered to be a vector quantity with direction perpendicular to the current loop in the right-hand-rule direction. The torque is given by

$$\tau = \mu \ \ x \ \ B$$

As seen in the geometry of a current loop, this torque tends to line up the magnetic moment with the magnetic field B, so this represents its lowest energy configuration. The potential energy associated with the magnetic moment is

$U(\theta) = -\mu \cdot B$ theta is the angle (dot product)

so that the difference in energy between aligned and anti-aligned is

$\Delta U = 2\mu B$ 180 degrees out of phase

These relationships for a finite current loop extend to the magnetic dipoles of electron orbits and to the intrinsic magnetic moment associated with electron spin. Also important are nuclear magnetic moments.

http://www.physicsclassroom.com/class/sound/u11l4d.cfm

the planck constant can be thought of as a string of length h vibrating at a frequency f.

As spin s=(n/2) increases in angular momentum in creates the nth harmonic of the base frequency.

These harmonics have a wavelength 2h/n and a frequency n times the base frequency.

The amplitude of the spin angular momentum cannot exceed the base amplitude by a ratio of ((n+1)/2)?

One could create a reduced divisor summatory function by subtracting the count of all trivial divisors 2*n*-1 and relating it to the prime counting function in the following way:

d(*n*) = divisor function, d(p)=2 for all primes p

SUM[d(*k*)] where 1<=k<=n = divisor summatory function

= D(*n)*

= count of divisors

pi(*n*) = count of prime numbers

n-1 = count of prime and composite integers, 1 is neither prime nor composite

2n-1 = count of trivial divisors

n-1–pi(*n*) = count of composite numbers

$D(n)-2n-1$ = reduced divisor summatory function

= count of non-trivial divisors

$D(n)-2(n-1-2\mathrm{pi}(n))$

http://en.wikipedia.org/wiki/Prime_number_theorem

http://en.wikipedia.org/wiki/Riemann_zeta_function

http://demonstrations.wolfram.com/HowTheZerosOfTheZetaFunctionPredictTheDistributionOfPrimes/

http://en.wikipedia.org/wiki/Divisor_summatory_function

Wavefunction Contexts

If the Electron Is a Wave, What is Waving?